Graph Theory and Its Applications

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1 Introduction

In this paper we will discuss how problems like Page ranking and finding the shortest paths can be solved by using Graph Theory. At its core, graph theory is the study of graphs as mathematical structures. In our paper, we will first cover Graph Theory as a broad topic. Then we will move on to Linear Algebra. Linear Algebra is the study of matrices. We will apply the skills discussed in these two sections to Dijkstra Algorithms which cover how to find the shortest paths in graphs. Finally, we will present PageRank where we will demonstrate how to rank pages based on their importance.

2 Graph Theory

In this section we will explain what a graph is as well as the different properties of a graph such as degrees, trails, vertices, and edges.

2.1 Definition of a graph

A graph is a collection of vertices and edges. Vertices can be thought of as dots that are connected by edges. For the purpose of this paper, we will assume that the graph has at most one edge between any two vertices. Graphs can be very resourceful tools used in real life in order to help people see where they can go and the different routes they can take to get there. There are various examples of these graphs, but for now we will use Königsberg Bridge Problem as an example. This problem consists of two islands in which seven bridges connect them and other various islands. The problem states that we can walk through the edges only once and we have to end up at the same place we started. However, we realized that it is impossible to go through every bridge exactly once and end up where we started. The Königsberg Bridge problem can be represented in a graph where the edges can be trans-versed either way, making it an undirected graph. An undirected graph is a graph that does not contain any arrows on its edges, indicating which way to go. A directed graph, on the other hand, is a graph in which its edges contain arrows indicating which way to go.

2.2 Properties of graph

In this section we will cover key properties of a graph. There are two main properties of a graph: degrees and walks. The degrees of a graph represents the number of vertices that a particular vertex is connected to. For example, if vertex 1 is connected to vertex 2 and 3, then vertex 1 would have a degree of 2. The walks in a graph refers to the different paths that can be taken to start at one vertex and end at another vertex or that same vertex. For example if one can go from vertex 1 to 2 then back to 1 or vertex 1 to 3 and back to 1, that means that there are two length two walks that start at vertex 1 and end at vertex 1. Figure 5 shows an example graph with walks and vertices' degrees labelled.

Definition 2.1. This is the definition

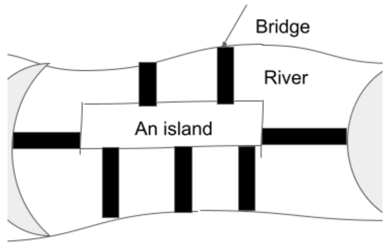

Figure 1: Königsberg Bridge

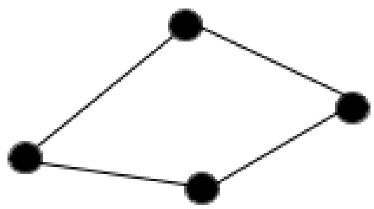

Figure 2: An Undirected Graph

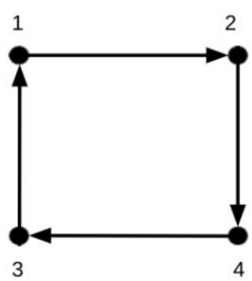

Figure 3: A directed Graph

2.3 Eulerian trail

Another property of a graph is a trail. A trail is a series vertices $v_1, \ldots, v_n$, where v_i, v_{i+1} are connected by an edge. A Eulerian trail contains every edge of the graph exactly once and is closed, if and only if, all vertices in the graph have even degrees.

Theorem 2.1. *A connected graph G has a closed Eulerian trail if and only all vertices of graph G has even degrees.*

3 Linear Algebra

Linear Algebra is a powerful tool that allows us to analyze graphs. In this section we will explain what a matrix is, what mathematical operations can be performed with matrices, and the different characteristics

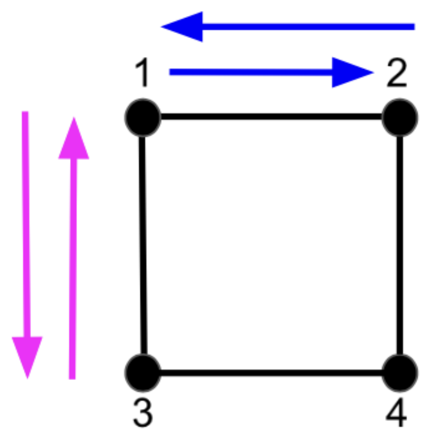

Figure 4: An image of Graph A that shows the degree of a vertex 1 as well as the amount of length two walks that start and end at vertex 1. Because Vertex 1 is connected to two vertices it has an edge of 2. There are also 2 two length walks that start and end at vertex 1.

of a matrix.

3.1 Definition of a matrix

Graphs and matrices are closely related to each other. A matrix is a set of numbers arranged in rows and columns so as to form a rectangular array. Some matrices can provide valuable information about graphs like how many vertices are connected, how many walks there might be between 2 vertices, and more. We will cover how to find the number of vertices connected to each other, as well as how many walks there might be between 2 vertices, and more, further ahead.

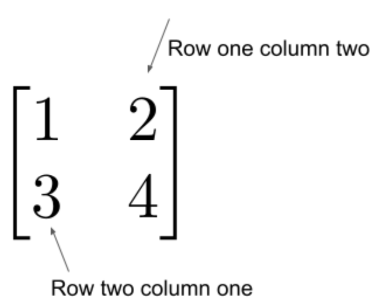

Figure 5: This is a Matrix

3.1.1 Matrix operations

Now that we have defined a matrix, we will present basic operations with them. There are three types of Matrix operations that we will be covering in our paper. One of the operations that will be covered is Addition. While the order of the matrices may not matter while adding matrices, both matrices need to have the same number of rows and columns. Once it is confirmed that both matrices have the same number of rows and columns, the corresponding entries can be added. The matrices below show a demonstration of how 2 2 x 2 matrices would be added.

This is an example of matrix addition:

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 5 & 3 \\ 7 & 2 \end{bmatrix} = \begin{bmatrix} 6 & 5 \\ 10 & 6 \end{bmatrix}$$

Another type of matrix operation is matrix multiplication by a constant. This operation is analogous to the multiplication of a number in front of an expression in parentheses, using the distributive property. As we can see in this example, in constant matrix multiplication, each entry in the matrix is multiplied by 2 to find the final product.

This is an example of constant by matrix multiplication:

$$2 * \begin{bmatrix} 5 & 3 \\ 7 & 2 \end{bmatrix} = \begin{bmatrix} 10 & 6 \\ 14 & 4 \end{bmatrix}$$

Finally, the remaining matrix operation is multiplication of two (or more) matrices with each other. When multiplying a matrix by another matrix the number of rows in the first matrix must be equal to the number of columns in the second matrix or else the two matrices can not be multiplied. Once it is clear that this requirement is met, the corresponding entry for column A of the first matrix has to be multiplied by the corresponding entry of row A of the second matrix. Then the products of multiplying all the entries from column A of the first matrix and row A of the second matrix would be added to find the corresponding entry for the product. This process would have to be continued until each column of the first Matrix has been multiplied by each row of the second matrix. As this example demonstrates, this is how it would look.

This is an example of a matrix by matrix multiplication:

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} * \begin{bmatrix} 5 & 3 \\ 7 & 2 \end{bmatrix} = \begin{bmatrix} 19 & 7 \\ 43 & 17 \end{bmatrix}$$

3.2 Determinants and Transpose

We will now define a property called Determinant for matrices. Determinant is a function that has a matrix as an input and a number as an output. For example, $det([4]) = 4$. When finding the determinant of a one by one matrix with a single entry, the determinant will always be the same as the input number. We will now explain how to find the determinant of larger matrices. To find the determinant of a 2 x 2 matrix we will need to do ad-bc, a being in row one column one, b being in row one column two, c being in row two column one, and d being in row 2 column 2. The next determinant we will find is of a 3 x 3 matrix. For this kind of problem a label must be placed on top of the first row as +, -, +. A + sign means the sign of the number stays the same, but a - sign means the sign of the number must be changed. Once the top row is labeled one can move on to finding the determinants of the numbers. To do this, start with the first number of the matrix which is the first number in column one and row one. To find the determinant of this number, the row and column that the number is in must be ignored. After that, a 2 x 2 matrix will be achieved which is needed to do $ad - bc$ which is the formula to find the determinant, as covered said before. This process mist be repeated two more times to find the determinants of the numbers in row one column two and row one column three. Finally, in order to find the determinant of a 4 x 4 matrix the same methods would be used, which first means labeling the first row with +, -, +, and -. After doing so, the rows and columns that the numbers are in must be cancelled out as one continues to solve. Doing this will result in a 3 x 3 matrix, which we have covered how to solve above. Although one can use the methods discussed above to solve for determinants, one can also use Theorem 3.1 which easily allows one to solve for any matrix of any size.

Theorem 3.1. *For any square matrix A, switch two neighboring rows to get A^I.* $\det(A) = -\det A^I$

A Transpose of a matrix is when the columns can be reflected to become rows. When a transpose is applied twice, the result is the original matrix. We write this as, Transpose of A = A^T

$$\begin{aligned}
&\det\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \\
=&a\det\begin{bmatrix} e & f \\ h & i \end{bmatrix} - b\det\begin{bmatrix} d & f \\ g & i \end{bmatrix} + c\det\begin{bmatrix} d & e \\ g & h \end{bmatrix} \\
=&a(e(i) - h(f)) - b(d(i) - g(f)) + c(d(h) - g(e)) \\
=&x
\end{aligned}$$

$$\det\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = 4(1) - 2(3) = -2$$

$$\begin{aligned}
&\det\begin{bmatrix} 3 & 5 & 3 \\ 2 & 7 & 1 \\ 4 & 4 & 2 \end{bmatrix} \\
=&3\det\begin{bmatrix} 7 & 1 \\ 4 & 2 \end{bmatrix} - 5\det\begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix} + 3\det\begin{bmatrix} 2 & 7 \\ 4 & 4 \end{bmatrix} \\
=&3(7(2) - 4(1)) - 5(2(2) - 4(1)) + 3(2(4) - 7(4)) \\
=&-30
\end{aligned}$$

$$\begin{aligned}
&\det\begin{bmatrix} 1 & 2 & 5 & 2 \\ 3 & 4 & 1 & 3 \\ 2 & 3 & 5 & 2 \\ 6 & 4 & 1 & 3 \end{bmatrix} \\
=&1\det\begin{bmatrix} 4 & 1 & 3 \\ 3 & 5 & 2 \\ 4 & 1 & 3 \end{bmatrix} - 2\det\begin{bmatrix} 3 & 1 & 3 \\ 2 & 5 & 2 \\ 6 & 1 & 3 \end{bmatrix} + 5\det\begin{bmatrix} 3 & 4 & 3 \\ 2 & 3 & 2 \\ 6 & 4 & 3 \end{bmatrix} - 2\det\begin{bmatrix} 3 & 4 & 1 \\ 2 & 3 & 5 \\ 6 & 4 & 1 \end{bmatrix}
\end{aligned}$$

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$$

Figure 6: The Transpose of a Matrix

4 Linear Algebra and Graph Theory

In this section we will talk about specific matrices and what these matrices can tell us about a graph.

4.1 Adjacency matrix

In this section we will cover the definition and uses of an adjacency matrix.

Definition 4.1. An adjacency matrix is a matrix of 0's and 1's based on whether or not two vertices have an edge between each other.

An adjacency matrix is a matrix of 0's and 1's based on whether or not two vertices have an edge between each other. If two vertices are connected to one another, the number 1 is inserted at the corresponding entry of the matrix. For example the corresponding entries for vertex 1 and 2 in the adjacency matrix is the first row, second column entry or the second row first column entry. If there is an edge between these two vertices, then a 1 can be placed in both of those entries. However, if two vertices are not connected, the number 0 can be used. For example, if someone were to reference row 1, column 4 and see the number 0, that means that there is no edge between vertex 4 and vertex 1.

Creating an adjacency matrix can also help determine the number of length k, k being any positive integer, walks from one vertex to another. For example, if the adjacency matrix is multiplied by itself once, which can be represented as $(A)^2$, and row 2 column 2 is referenced and the number 2 is there that means that there are 2 length two walks that start and end at vertex 2. In order to find the total number of length k walks in the entire graph, the adjacency matrix would have to multiplied by itself k times, which can be represented as $(A)^k$. Then that product would have to be multiplied by a vertical matrix of 1's. Finally, that product would have to be multiplied by a matrix of horizontal 1's in order to find the total number of length k walks in the graph. For example, in order to determine the number of length two walks in the graph, the adjacency matrix would be squared and then the rest of the steps would be followed normally The figures below show a labeled edge between two vertices in Graph A, the adjacency matrix of graph A, and how to find the number of length two walks in Graph A.

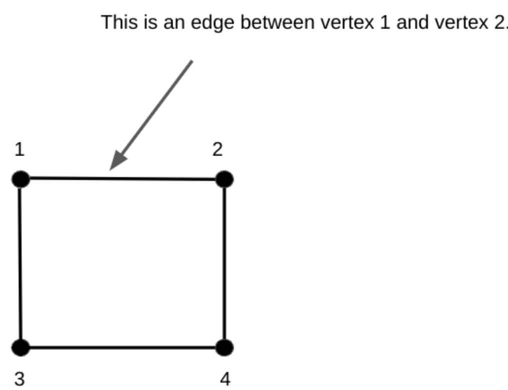

Figure 7: an image of graph A with a labelled edge between vertex 1 and vertex 2.

$$\begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

This is the adjacency matrix for Graph A which is important in finding the number of length k walks in Graph A.

$$\begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} * \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 & 2 \\ 0 & 2 & 2 & 0 \\ 0 & 2 & 2 & 0 \\ 2 & 0 & 0 & 2 \end{bmatrix}$$

This is the first step that must be taken to find the number of length two walks. The adjacency matrix is being squared.

$$\begin{bmatrix} 2 & 0 & 0 & 2 \\ 0 & 2 & 2 & 0 \\ 0 & 2 & 2 & 0 \\ 2 & 0 & 0 & 2 \end{bmatrix} * \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \\ 4 \\ 4 \end{bmatrix}$$

This is the second step of finding the number of length 2 walks in Graph A. The squared adjacency matrix or $(A)^2$ is being multiplied by a vertical matrix of 1's.

$$\begin{bmatrix} 4 \\ 4 \\ 4 \\ 4 \end{bmatrix} * \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 16 \end{bmatrix}$$

This is the last step of finding the number of length 2 walks in Graph A. The vertical matrix of 4's is being multiplied by the horizontal matrix of 1's to get a matrix of 16.

These matrices show the process used to find the number of length k walks in a graph. This example shows the process used to find the number of length two walks in graph A. First the adjacency matrix is multiplied by itself to get a matrix of 2's and 0's. Then that product is multiplied by a vertical matrix of 1's to get a matrix product of vertical 4's. Then the vertical matrix of 4's is multiplied by a horizontal matrix of 1's to get a matrix of 16. This means that there are 16 length 2 walks in graph A.

4.2 Incidence matrix

In this section we will cover the definition of an incidence matrix.

Definition 4.2. An incidence matrix is the matrix of a directed graph, or a graph with directional edges.

An Incidence matrix is the matrix of a directed graph, or a graph with directional edges. Each column in an incidence matrix represents an edge between two vertices. The incidence matrix is made up of 1's, -1's, and 0's. The number 1 represents leaving a vertex. The number -1 represents arriving at a vertex. The number 0 means that the vertex is not involved. The figure below is an example of an incidence matrix.

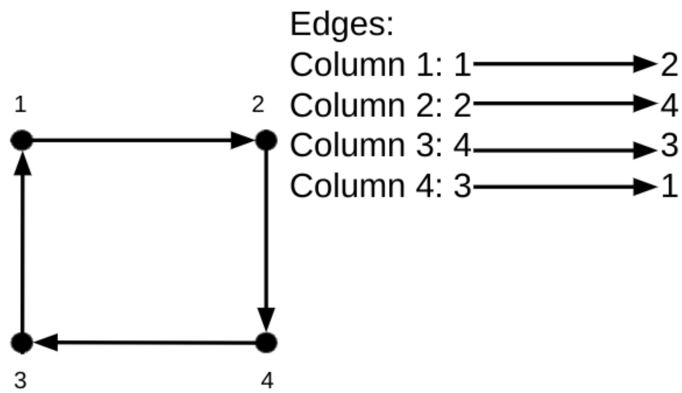

Figure 8: An image of the directed version of Graph A where each edge represents a particular column in the incidence matrix.

$$\begin{bmatrix} 1 & 0 & 0 & -1 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & -1 & 1 & 0 \end{bmatrix}$$

This is the Incidence matrix for the direction version of Graph A. Again, 1 represents leaving a vertex, -1 represents arriving at a vertex, and 0 means that the vertex is not involved.

4.3 Laplacian matrix

In this section we will cover the two formulas used to find the Laplacian Matrix.

Definition 4.3. Laplacian Matrix can be defined as such: $L_0(L_0)^T$ where L_0 represents the Incidence matrix and L_0^T represents the transpose of an Incidence matrix.

If one wishes to find the Laplacian matrix of an undirected graph, they can assign directions to the edges in the graph. The Laplacian Matrix can also be found by using a much simpler formula known as $D - adj(G)$ In this formula, D is a matrix where diagonal entries are degrees of vertices and all other entries are 0 and $adj(G)$ represents the Adjacency matrix.

$$\begin{bmatrix} 1 & 0 & 0 & -1 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & -1 & 1 & 0 \end{bmatrix} * \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & -1 & 1 \\ -1 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & -1 & -1 & 0 \\ -1 & 2 & 0 & -1 \\ -1 & 0 & 2 & -1 \\ 0 & -1 & -1 & 2 \end{bmatrix}$$

The Incidence matrix or L_0 is being multiplied by the transpose of the Incidence matrix or $(L_0)^T$.

This equation shows the first equation that could have been used to find the Laplacian matrix. In this equation, the first matrix or the Incidence matrix which is represented by L_0, is multiplied by the the transpose of the Incidence matrix, or L_0^T. The product of these two factors is the Laplacian matrix or $L_0(L_0)^T$.

We also could have done...

$$\begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} - \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & -1 & -1 & 0 \\ -1 & 2 & 0 & -1 \\ -1 & 0 & 2 & -1 \\ 0 & -1 & -1 & 2 \end{bmatrix}$$

This shows the adjacency matrix or $adj(G)$ is being subtracted from the diagonal matrix or D.

This equation represents the other equation that could have been used to find the Laplacian matrix. In this equation, the Adjacency matrix, or $adj(G)$ is subtracted from D or the Diagonal matrix. The difference between these two matrices is equal to the Laplacian matrix as well.
Remark: The directions assigned to a graph do not make a difference once the final Laplacian matrix is found. While the Incidence matrix will be different based on how the edges are directed, once the Incidence matrix is multiplied by its transpose, the Laplacian matrix will be the same no matter what the Incidence matrix is.

5 Dijkstra Algorithm

5.1 The shortest path problem

The Dijkstra Algorithm is used to find the shortest path from a 'source node' to other nodes in the graph. A node is a place, person, or object and the thing that connects these nodes are the edges. The example we'll be using to demonstrate this today is the example the inventor himself, Dijkstra, used. This problem includes a weighted graph which is a graph that has some type of weight or cost on the edges of the graph, a weight being the length of a path. These weights or costs can be anything from distance or time, to anything that represents a connection between the nodes. Once these nodes have been visited, meaning that one has seen the path to take from the source node, which is going to be node 1 for this section, to another node, one can put a red mark next to the number to signal that. In this example there is a graph with four nodes and four weighted edges. We are trying to see what the shortest path is from node 1 to the other nodes which can be the distance, time, or anything that models the "connection" between the pair of nodes connected.

5.2 Dijkstra Algorithm Steps

To solve the problem above the first step is to see what the shortest path is from the source node to node 1, which are the same thing. Since the distance from the source node to the rest have not yet been established we will put infinity signs next to the nodes that are not visited. Once nodes have been visited and the shortest path has been discovered we can add them to the path. Because we are starting from

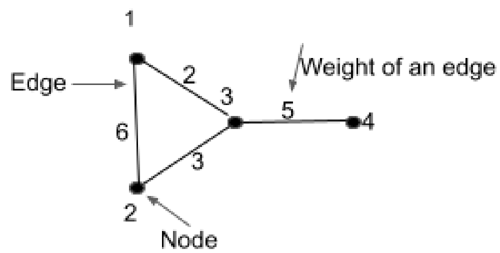

Figure 9: An image of a weighted graph

node 1 as our source node we can mark it as visited and included in the path. Then we check the length of the path of the nodes adjacent to the source node and write down the distance from the source node to its adjacent nodes. Some adjacent nodes are 1 and 2 with a weighted edge of 6. Because we have discovered the distance and the shortest way to reach node 2 from the source node, we can put 6 on node 2. Then we look at the node adjacent to node 2 and repeat the same steps as before. When finding the distance between two edges connecting two nodes one must add the weight/cost to find the distance between the source node and the adjacent nodes. If the distance from the source node to an adjacent node can be reached in different ways we will check to find the shortest path. If there are different results we will update it to the shortest path but if there are no differing results, we will not update it.

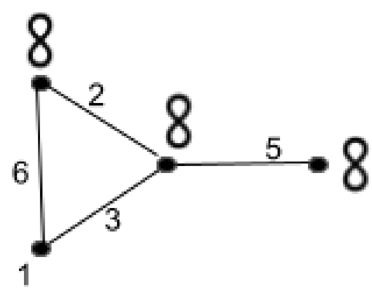

Figure 10: An image of a weighted graph with no shortest path's discovered

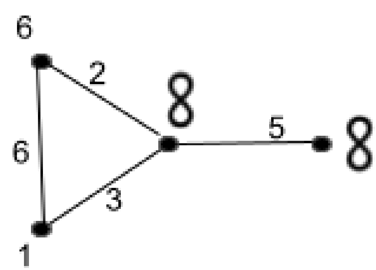

Figure 11: An image of a weighted graph with one shortest path discovered

6 PageRank

One of the many real world applications of Graph Theory is Page ranking or PageRank. In this section we sill talk about the history of PageRank and how it can be represented with a transition matrix and a directed graph.

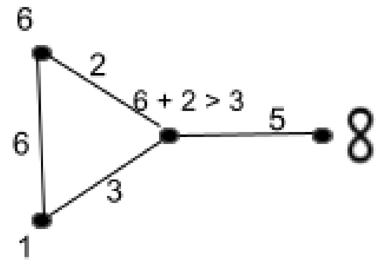

Figure 12: An image of a weighted graph with two shortest paths discovered

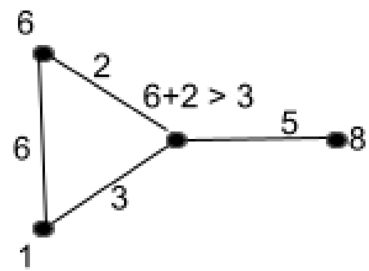

Figure 13: An image of a weighted graph with all shortest paths discovered

6.1 History of PageRank

Up until the early 90s, search engines used text based ranking systems to decide which pages were the most relevant and therefore the first displayed. This means that if a search was made about a common term like the internet, the first website that would appear is the website that says the word internet the most amount of times. Although this approach may seem sensible, it actually has many flaws. Just because a webpage more frequently contains a keyword does not necessarily mean that it is the most relevant to the question a web user could be trying to answer. The page ranking system, however, determines the relevance of a web page based on the other sources that it is linked to. This approach of ranking websites allows users to get the best possible source for the question they are trying to answer.

6.2 Graph Representation of the internet

The connections between the webpages can be represented by a directed graph. If a website has a link to another website then an arrow that is pointing towards the linked website can be drawn. If that linked website also has a link to the same website that it was linked to, there can also be an arrow pointing in the opposite direction. For example, if website 1 links to website 4 than an arrow can be drawn moving away from website 1. Moreover, if website 4 links to website 1, then an arrow can be drawn moving towards website 1.

6.3 PageRank Algorithm

As was briefly mentioned in the first section about page ranking, the relevance of websites is determined by the other websites that is linked to it rather than the amount of times a key word appears on the website. For example, if there are 4 different websites that are all about the same topic and all the other websites have links to the third website, it can be assumed that the third website is the most relevant source. Then if only two websites have links to website 1 and website 4, it can be assumed that both websites are the second most relevant sources. Lastly if only one website has a link to website 2, it can be assumed that website 2 is the least relevant source. The weight of each of these connections between websites can be determined based on how many websites each website has links to. This means that equal weight is assigned

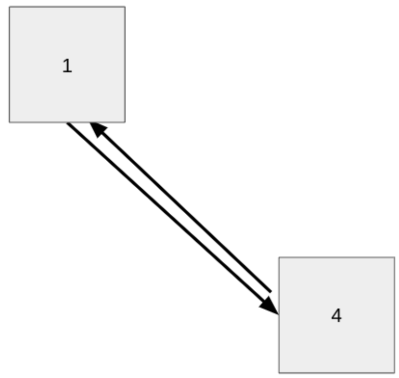

Figure 14: an example of how a connection can be represented between website 1 and website 4.

to each arrow leaving the same website and typically these weights are supposed to add up to 1. The connection between websites could be represented in a graph and a transition matrix like the figures below:

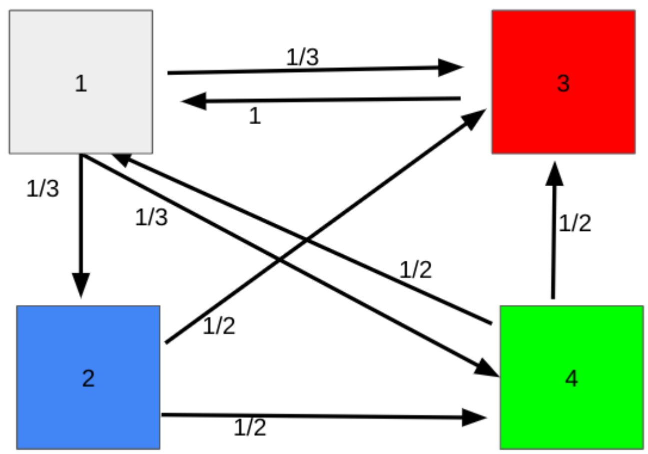

Figure 15: This is an image of the connection between 4 websites and an example of how these connections can be represented in a directed graph with weights assigned to each arrow.

$$\begin{bmatrix} 0 & 0 & 1 & \frac{1}{2} \\ \frac{1}{3} & 0 & 0 & 0 \\ \frac{1}{3} & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{2} & 0 & 0 \end{bmatrix}$$

Figure 16: This is the transition matrix that represents the connections between the 4 websites. The first row represents the weight of the arrows arriving at website 1 and the first column represents the weight of the arrows leaving website 1. The same can be said for website 2 about the second row and the second column, website 3 about the third row and the third column and website 4 for the fourth row and fourth column.